Group actions, geodesic loops, and symmetries of compact hyperbolic 3-manifolds.

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Abstract.

Compact hyperbolic 3-manifolds are used in cosmological models. Their topology is characterized by their homotopy group $\pi_1(M)$ whose elements multiply by path concatenation. The universal covering of the compact manifold M is the hyperbolic space H^3 or the hyperbolic ball B^3. They share with M a Riemannian metric of constant negative curvature and allow for the isometric action of the group $Sl(2, C)$. The homotopy group $\pi_1(M)$ acts as a uniform lattice $\Gamma(M)$ on B^3 and tesselates it by copies of M. Its elements g produce preimage and image points for geodesic sections on B^3 which by self-intersection form geodesic loops on M. For any fixed hyperbolic $g \in \Gamma$ we construct a continuous commutative two-parameter normalizer $N_g < Sl(2, C)$ and its orbit surfaces on B^3. The orbit surfaces classify sets of geodesic loops of equal length. We give general expressions for the length of geodesic loops and for the defect angle at the self-intersection on M in terms of the group parameters of g and orbit parameters on B^3. Geodesic loops of minimal length, given from the character $\chi(g)$, belong to a single orbit. These and only these minimal geodesic loops have vanishing defect angle and hence are smooth everywhere. The role of symmetries is illuminated by the example of the dodecahedral hyperbolic Weber-Seifert manifold M. $\Gamma(M)$ is normal in the hyperbolic Coxeter group with Coxeter diagram $\circ \overset{5}{—} \circ \overset{3}{—} \circ \overset{5}{—} \circ$. This leads to symmetry relations between geodesic loops.

1 Introduction.

Models of a closed cosmos with nontrivial topology were reviewed by Lachieze and Luminet in [6] and by Levin in [7]. The predictions from the models may be compared directly with astronomical data, compare Fagundes [2], [3], or with the autocorrelation of the observed cosmic mass density [6] pp. 200-201. A well-known example

is the dodecahedral hyperbolic manifold M due to Weber and Seifert [13], compare Best [1] and Thurston [12] pp 36-37. Closed hyperbolic manifolds can be ordered by their volume. Those of small volume due to Thurston and to Weeks have found particular attention in cosmology [7] p. 265. As pointed out in [7] p. 266, geodesic loops on the manifold M convey important information for the spacing of ghost images and for the autocorrelation of the mass density.
We use continuous and discrete groups for the analysis of geodesic loops. We consider a general compact hyperbolic manifold M whose universal covering is the hyperbolic space H^3 or hyperbolic ball $B^3 \sim H^3$. H^3 is equivalent to the coset space $SO^+_\uparrow(1,3,R)/SO(3,R)$, admits the isometric action of $SO^+_\uparrow(1,3,R)$ or of its universal covering group $Sl(2,C)$, and has constant negative curvature [14]. The Minkowski metric restricted to H^3 yields the notions of geodesics and their length, which carry over to M. The homotopy group $\pi_1(M)$ multiplies by path concatenation. Typically it is a finitely generated (and even finitely presented) infinite discrete group.
The uniform lattice $\Gamma(M)$ is isomorphic to $\pi_1(M)$, acts as a discrete subgroup of $Sl(2,C)$ without fixpoints on the universal covering manifold B^3, and generates a tesselation by copies of M. Pairs of points on different copies of M in B^3 are equivalent if there is an element g of $\Gamma(M)$ which has this pair as preimage and image. Equivalent points of B^3 are identified on M. Any geodesic section on B^3 between equivalent points when mapped to M forms a geodesic loop with self-intersection.
A classification of geodesic loops on M should deal with two aspects: (i) Find the variety of geodesic loops for a given fixed element $g \in \Gamma$. (ii) Compare geodesic loops for different elements of $\Gamma(M)$, leading to a length spectrum of geodesic loops. This requires an analysis of the elements of $\Gamma(M)$, taken as words in its generators. In what follows we consider mainly the aspect (i) for a general closed hyperbolic manifold M. In section 2 we briefly describe the groups, in section 3 the geometry of the hyperbolic space H^3 and hyperbolic ball B^3, and the group actions. In section 4 we develop the classification of geodesic loops, and in section 5 and 6 we illustrate symmetries on the compact hyperbolic Weber-Seifert manifold.

2 The Lorentz group $SO^+_\uparrow(1,3,R)$, its covering, and Weyl reflections.

The universal covering manifold for a compact hyperbolic manifold M is the hyperbolic space H^3, a homogeneous space under the Lorentz group $SO^+_\uparrow(1,3,R)$. In this section we collect results and notations for the action of this and related continuous groups. In the next sections they will be used for the discrete uniform lattice $\Gamma(M)$.

2.1 $Sl(2,C)$: Class and in-class structure, adjoint representation.

The universal covering of the proper time-preserving Lorentz group $SO^+_\uparrow(1,3,R)$ is the unimodular group $Sl(2,C)$. Both groups have 6 real parameters.
We omit from the discussion unipotent elements of $Sl(2,C)$ with a Jordan decomposition since such elements cannot appear in Γ by [9], theorem 4.1 (Kazhdan/Margulis), p. 79. Then all elements g of interest have class representative g_0 in diagonal form. Exponential parameters can be used to display g_0 as

$$g_0 = \left[\begin{array}{ll} \exp(c+i\gamma) & 0 \\ 0 & \exp(-(c+i\gamma)) \end{array}\right], \; -\infty < c < \infty, \; 0 \leq \gamma < 2\pi. \tag{1}$$

with two real class parameters $c,\ \gamma$. The elements eq. 1 belong to a subgroup H isomorphic to $SO(1,1,R) \times SO(2,R)$ but in diagonal, not in standard form. We call hyperbolic the elements eq. 1 and their conjugates which in addition obey $|c| > 0$. Given a general element $g \in Sl(2,C)$, we can determine its class parameters by use of the complex character or trace $\chi(g)$,

$$\frac{1}{2}\chi(g_0) = \frac{1}{2}Tr(g_0) = \cosh(c+i\gamma) = \cosh(c)\cos(\gamma) + i\sinh(c)\sin(\gamma). \tag{2}$$

We define the additional elements

$$\begin{aligned} g_1 &= g_1(a,\alpha) = \left[\begin{array}{ll} \cosh(a+i\alpha) & \sinh(a+i\alpha) \\ \sinh(a+i\alpha) & \cosh(a+i\alpha) \end{array}\right], \; (a,\alpha) \text{ real}, \qquad (3) \\ g_2 &= g_0(b,\beta) = \left[\begin{array}{ll} \exp(b+i\beta) & 0 \\ 0 & \exp(-(b+i\beta)) \end{array}\right], \; (b,\beta) \text{ real}. \end{aligned}$$

By use of complex Euler angles it can be shown that the products g_2g_1 parametrize the cosets $Sl(2,C)/H$. By $\sigma_1, \sigma_2, \sigma_3$ we denote the standard hermitian Pauli matrices and by e the 2×2 unit matrix. We claim: The general element g of the class with representative eq. 1 may be written as

$$\begin{aligned} g &= (g_2g_1)g_0(g_1g_2)^{-1}, \qquad (4) \\ &= \cosh(c+i\gamma) + \sinh(c+i\gamma)\sum_{i=1}^{3}\eta_i\sigma_i, \; \sum_{i=1}^{3}(\eta_i)^2 = 1, \\ \left[\begin{array}{l} \eta_1 \\ \eta_2 \\ \eta_3 \end{array}\right] &= Ad(g_2g_1)\left[\begin{array}{l} 0 \\ 0 \\ 1 \end{array}\right], \\ Ad(g_2g_1) &= \left[\begin{array}{lll} \cosh(2(b+i\beta)) & -i\sinh(2(b+i\beta)) & 0 \\ i\sinh(2(b+i\beta)) & \cosh(2(b+i\beta)) & 0 \\ 0 & 0 & 1 \end{array}\right] \end{aligned}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh(2(a+i\alpha)) & -i\sinh(2(a+i\alpha)) \\ 0 & i\sinh(2(a+i\alpha)) & \cosh(2(a+i\alpha)) \end{bmatrix}.$$

The trace is not affected by the conjugation in eq. 4 and so the class parameters c, γ are obtained from the character $\chi(g) = \chi(g_0)$ as in eq. 2. We call a, α, b, β the 4 in-class parameters of the class. For $a = b = c = 0$, eq. 4 becomes a variant of the familiar parametrization of $SU(2)$.
The adjoint representation $Ad(Sl(2,C))$ is formed by the complex orthogonal group $SO(3,C)$. For the elements $g_2, g_1 \in Sl(2,C)$, the adjoint representation is generated by the matrices $Ad(g_2),\ Ad(g_1)$ in eq. 3. It is expressed by two complex rotations which extend the two real rotations used with the adjoint representation of $SU(2)$. For $a = b = c = 0$, the adjoint representation becomes equivalent to $Ad(SU(2)) \sim SO(3,R)$. The homomorphism $Sl(2,C) \rightarrow Ad(Sl(2,C))$ is two-to-one with $Ad(g) = Ad(-g)$.

2.2 The proper time-preserving Lorentz group and Weyl reflections.

We obtain in the usual fashion the two-to-one homomorphism from $Sl(2,C)$ to the Lorentz group with transformations $L(g) \in S0^{+}_{\uparrow}(1,3,R)$. We use in Minkowski space $M(1,3)$ the coordinates (x_0, x_1, x_2, x_3). The scalar product is taken as

$$\langle x, y\rangle = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3. \tag{5}$$

We introduce the 2×2 hermitian matrix

$$\tilde{x} = x_0 e + \sum_{i=1}^{3} x_i\sigma_i. \tag{6}$$

with $det(\tilde{x}) = \langle x, x\rangle$. The linear Lorentz action $L(g) = L(-g)$ of $Sl(2,C)$ on $M(1,3)$ is

$$\begin{aligned} g \in Sl(2,C) : \tilde{x} \rightarrow \tilde{x}' &= g\tilde{x}g^{\dagger}, \qquad (7)\\ x'_{\mu} &= \sum_{0}^{3} L_{\mu\nu}(g)x_{\nu}. \end{aligned}$$

In particular one finds from eqs. 1, 3, 7

$$L(g_0(c,\gamma)) := \begin{bmatrix} \cosh(2c) & 0 & 0 & \sinh(2c) \\ 0 & \cos(2\gamma) & \sin(2\gamma) & 0 \\ 0 & -\sin(2\gamma) & \cos(2\gamma) & 0 \\ \sinh(2c) & 0 & 0 & \cosh(2c) \end{bmatrix}, \tag{8}$$

$$
\begin{aligned}
L(g_1(a,\alpha)) &:= \begin{bmatrix} \cosh(2a) & \sinh(2a) & 0 & 0 \\ \sinh(2a) & \cosh(2a) & 0 & 0 \\ 0 & 0 & \cos(2\alpha) & \sin(2\alpha) \\ 0 & 0 & -\sin(2\alpha) & \cos(2\alpha) \end{bmatrix}, \\
L(g_2) &:= L(g_0(b,\beta)).
\end{aligned}
$$

The Lorentz transformations $L(g_0(c,\gamma))$ for hyperbolic g_0 with $|c| > 0$ as defined after eq. 1 can have no fixpoint on the coset space H^3.
Given a vector $k \in M(1,3)$, $\langle k,k\rangle \neq 0$, we define the Weyl reflection operator $W_k : M(1,3) \to M(1,3)$ by

$$
W_k : x \to x - 2\frac{\langle k,x\rangle}{\langle k,k\rangle}\, k. \tag{9}
$$

The Weyl reflection preserves any point of the reflection hyperplane $\langle x : \ \langle k,x\rangle = 0\rangle$. Weyl operators are isometries with respect to the metric of $M(1,3)$. They allow to extend the proper time-preserving Lorentz group by space reflections. In particular for $k = e_2 = (0,0,1,0), \langle k,k\rangle = -1$, the Weyl reflection W_{e_2} inverts only the coordinate x_2. Under Lorentz transformations one easily proves the conjugation law

$$
L(g)W_kL(g^{-1}) = W_{L(g)k}. \tag{10}
$$

3 The hyperbolic space H^3 and ball B^3.

The hyperbolic space H^3 arises as the universal covering space of compact hyperbolic manifolds. This universal covering space plays a key role in the analysis of geodesic loops in sections 4-6. For details on the hyperbolic space and ball we refer to Ratcliffe [10] pp. 56-104 and pp. 127-135 respectively.
The hyperbolic space is the coset space $H^3 := SO^+_\uparrow(1,3,R)/SO(3,R)$. In $M(1,3)$ its points form the hyperboloid

$$
H^3 = \langle x|\langle x,x\rangle = 1, x_0 \geq 1\rangle. \tag{11}
$$

We follow [10] up to a sign and rewrite the scalar product eq. 5 in $M(1,3)$ as

$$
\begin{aligned}
\langle x,y\rangle &= x_0y_0 - (x,y), \\
(x,y) &:= x_1y_1 + x_2y_2 + x_3y_3.
\end{aligned} \tag{12}
$$

Given two points $x, y \in H^3$, we take their scalar product as the restriction of the scalar product $\langle x,y\rangle$ from $M(1,3)$ to H^3. The restricted scalar product on H^3 is positive definite. H^3 with this metric can be shown to be a space of constant negative curvature [14] .

The model of the conformal ball B^3 for H^3 is obtained from the points of the hyperboloid eq. 11 by the fractional linear map to a Euclidean space E^3,

$$
\begin{aligned}
x_i \rightarrow \xi_i &= \frac{x_i}{1+x_0},\ i=1,2,3,\ (\xi,\xi)<1, \\
x_i &= \frac{2\xi_i}{1-(\xi,\xi)},\ i=1,2,3.
\end{aligned}
\tag{13}
$$

Although all points of H^3 from eq. 13 map bijectively into points of B^3, we shall need points in E^3 but outside B^3 to characterize its symmetries. We shall use the scalar product $(,)$ with respect to the space-like components of vectors from $M(1,3)$, for vectors in B^3, and in E^3 embedding B^3. Consider a space-like vector $k \in M(1,3),\ k_0 \neq 0$, and a point $x : \langle x, x\rangle = 1$ of H^3. Using for x the coordinates eq. 13 on B^3 one finds

$$
\begin{aligned}
\langle k, x\rangle &= k_0(1+x_0)\frac{1}{2}(1+(\xi,\xi)-2(q,\xi)), \\
k &\rightarrow q = k_0^{-1}(k_1,k_2,k_3).
\end{aligned}
\tag{14}
$$

1 Lemma: The intersection of the hyperplane $\langle k, x\rangle = 0,\ k_0 \neq 0$ in $M(1,3)$ with the hyperboloid H^3 in the conformal ball model becomes the intersection of a Möbius sphere $S^2(q,R)$ of center $q, (q,q) > 1$ and radius R with $B^3 \subset E^3$,

$$
\begin{aligned}
(\xi-q,\xi-q)-R^2 &= 0, \\
q &= k_0^{-1}(k_1,k_2,k_3),\ R=\sqrt{(q,q)-1}.
\end{aligned}
\tag{15}
$$

The Möbius sphere has orthogonal intersections with ∂B^3.
Proof: It suffices to rewrite part of eq. 14 as

$$
(1+(\xi,\xi)-2(q,\xi)) = (\xi-q,\xi-q)-(q,q)+1 = (\xi-q,\xi-q)-R^2. \tag{16}
$$

The condition $R^2 = (q,q)-1$ assures that the Möbius sphere $S^2(q,R)$ has orthogonal intersections with the surface $\partial B^3 :\ (\xi,\xi) = 1$. The sphere $S^2(q,R)$ separates the points of B^3 into two disjoint parts. In case of a vector $k : (0,k_1,k_2,k_3)$ when q in eq. 15 is not well-defined we replace the sphere in E^3 by a plane through the origin and perpendicular to (k_1,k_2,k_3).
Given a Weyl reflection eq. 9 with space-like Weyl vector k one finds
2 Lemma: A Weyl reflection W_k eq. 9 with Weyl vector k space-like in E_3 becomes a Mbius inversion in the sphere $S^2(q,R)$ with center and radius (q,R) from eq. 15. This Mbius inversion sends B^3 into B^3 and transforms points inside and outside of $S^2(q,R)\cap B^3$ into one another. The Mbius inversion is conformal, angles between geodesics are preserved.
Proof: We can write the action of the Weyl operator in B^3 from eqs. 9, 13 as

$$
\xi_i \rightarrow \kappa_i = \frac{\xi_i+(1+(\xi,\xi)-2(q,\xi))k_0^2q_i}{1+(1+(\xi,\xi)-2(q,\xi))k_0^2},\ i=1,2,3. \tag{17}
$$

If in E^3 we transform to new coordinates with respect to the center q of $S^2(q, R)$ by putting in eq. 13 $\xi_i = q_i + u_i,\ \kappa_i = q_i + v_i$, we get the Weyl reflection eq. 9 on B^3 in the form

$$u_i \to v_i = u_i \frac{R^2}{(u,u)},\ (v,v)(u,u) = R^4. \tag{18}$$

This is the standard form of a Mbius inversion in the sphere $S^2(q, R)$, compare [10] pp.109-11, and therefore is conformal.

4 Geodesics on H^3 and B^3.

As pointed out in the introduction, for any geodesic loop on a fixed compact hyperbolic manifold M there is a unique element $g \in \Gamma(M)$ acting on H^3. We wish to characterize the variety of geodesic loops associated with a fixed g. To this purpose we construct the continuous normalizer of $g \in SL(2, C)$ and its orbits on H^3. The orbits classify geodesic loops by length and direction and yield explicit expressions for them. A geodesic which closes on M must intersect itself. From H^3 we compute the defect angle at the self-intersection. For given g, there is a unique set of geodesic loops which have minimum length and are smooth, i.e. have vanishing defect angle.

4.1 Two-parameter normalizer subgroups of $Sl(2, C)$ and orbit surfaces.

With a discrete element g of $Sl(2, C)$ we associate a continuous group which allows to classify geodesic sections related to g.
3 Def: Consider a discrete element $g \in Sl(2, C)$ in diagonal form, $g = g_0(c, \gamma)$, eq. 1 with fixed exponential parameters c, γ. Define the group N_{g_0} with elements

$$h(\lambda, \phi) := g_0(\lambda c/2, \lambda\gamma/2 + \phi),\ -\infty < \lambda < \infty,\ 0 \le \phi < 2\pi. \tag{19}$$

This is the two-parameter commutative normalizer N_{g_0} (which coincides with the centralizer) of the group generated by g_0. Its elements must commute with g_0, eq. 1. N_{g_0} is isomorphic to $SO(1, 1, R) \times SO(2, R)$. With $L(g_0(\lambda c/2, \lambda\gamma/2 + \phi))$ we represent N_{g_0} by Lorentz transformations. The reasons for our choice of parameters will appear from the actions.
Turn to the action of N_{g_0} on $M(1, 3), H^3, B^3$. On H^3 we can choose the hyperplane $\langle e_3, x\rangle = 0$ for the orbit representatives on H^3 under N_{g_0}. We incorporate the second parameter ϕ of N_{g_0} into the representatives and write them as

$$x(\rho, \phi) = (\cosh(2\rho), \sinh(2\rho)\cos(2\phi), \sinh(2\rho)\sin(2\phi), 0),\ 0 \le \rho < \infty, 0 \le 2\phi < 2\pi. \tag{20}$$

The points on H^3, B^3 under the action of N_{g_0} we call orbit surfaces. Each orbit surface is determined by a single value of ρ. We call orbit lines the points from the action of N_{g_0} on B^3 for varying λ and fixed ϕ. The full orbit surface is obtained by rotating this orbit line with the angle ϕ.

4 Lemma: Preimages on a fixed orbit surfaces under g_0 are mapped into images on the same orbit surface. The relative geodesic distance between preimage and image point is independent of the starting point on the orbit surface.

Proof: The first part follows from the fact that N_{g_0} commutes with g_0. Any point on an orbit surface may be written as $L(g_0(\lambda c/2, \lambda\gamma/2))x(\rho, \phi)$. With $g_0 = g_0(c, \gamma)$ we obtain from the scalar product for the geodesic distance between preimage and image points under $L(g_0(c, \gamma))$

$$\begin{aligned}&\langle L(g_0(-\lambda c/2, -\lambda\gamma/2))x(\rho, \phi), L(g_0(c, \gamma))L(g_0(-\lambda c/2, -\lambda\gamma/2))x(\rho, \phi)\rangle \\ &= \langle x(\rho, \phi), L(g_0(c, \gamma))x(\rho, \phi)\rangle \ = \langle x(\rho, 0), L(g_0(c, \gamma))x(\rho, 0)\rangle.\end{aligned} \tag{21}$$

The Lorentz invariance of the scalar product and the commutativity of N_{g_0} are crucial for this result.

We choose as specific preimages on H^3 for the orbit lines the points $L(g_0(-c/2, -\gamma/2))x(\rho, \phi)$. The images then are of the form $L(g_0(c/2, \gamma/2))x(\rho, \phi)$. Preimages and images correspond to parameter values $\lambda = \mp 1$ in eq. 20 respectively. For fixed λ, all orbit lines pass the hyperplane $\langle k, x\rangle = 0,\ k = L(g_0(\lambda c/2, \lambda\gamma/2))e_3$ of H^3.

On H^3 the orbit lines take the form

$$L(g_0(\lambda c/2, \lambda\gamma/2))x(\rho, \phi) = \begin{bmatrix} \cosh(2\rho)\cosh(\lambda c) \\ \sinh(2\rho)\cos(2\phi - \lambda\gamma) \\ \sinh(2\rho)\sin(2\phi - \lambda\gamma) \\ \cosh(2\rho)\sinh(\lambda c) \end{bmatrix}. \tag{22}$$

We compute the orbit lines in B^3 and find by applying eq. 13 to eq. 22

$$\xi(\lambda) = (1 + \cosh(2\rho)\cosh(\lambda c))^{-1} \begin{bmatrix} \sinh(2\rho)\cos(2\phi - \lambda\gamma) \\ \sinh(2\rho)\sin(2\phi - \lambda\gamma) \\ \cosh(2\rho)\sinh(\lambda c) \end{bmatrix}. \tag{23}$$

For fixed λ, the orbit lines $\xi(\lambda)$ intersect the Möbius sphere $S^2(q, R)$ with

$$q(\lambda) = (0, 0, \mathrm{cotanh}(\lambda c)),\ R(\lambda) = (\sinh(\lambda c))^{-1}. \tag{24}$$

Since the action of $Sl(2, C)$ is conformal we have

5 Lemma: The orbit lines on B^3 for fixed parameter λ intersect the Möbius sphere eq. 24. The angle between tangents to orbit lines and the normal of the Möbius sphere depends on the starting value of ρ but is independent of λ, ϕ.

All orbit lines for $\lambda = 0$ cross the plane $\xi_3 = 0$. This choice allows for a simple visualization of the orbit lines and geodesics in Fig. 1. The full orbit surfaces are obtained by rotating the orbit lines to any angle ϕ around the 3-axis.

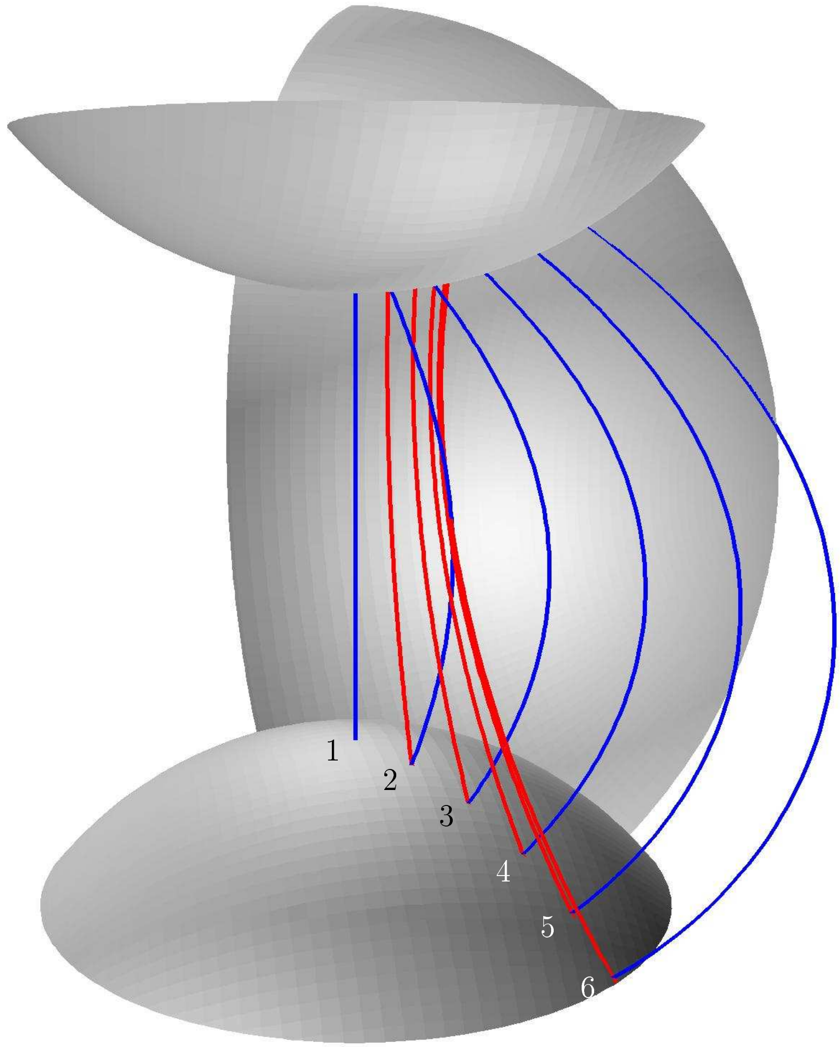

Fig. 1. A sector of area π from the boundary of B^3 and its intersection with two Möbius spheres $S^2(q, R)$. The chosen element $g = C_1 \in \Gamma(M)$ is a generator of Γ for the compact Weber-Seifert manifold. It maps the lower into the upper Möbius sphere. Shown are six orbit lines $1, \ldots, 6$ for this map, and six geodesic sections between preimages and images under $g = C_1$. The orbit surface arises from each orbit line by a rotation around the vertical 3-axis. Only the straight orbit line 1 coincides with the shortest geodesic section 1 under C_1.

4.2 Geodesic lines on B^3.

6 Lemma: The geodesic connecting two points $\xi, \eta,\ \xi \neq \eta$ on B^3 is the section between the points $\langle \xi, \eta \rangle$ on the circle $S^1(q, R)$ which has two perpendicular inter-

sections with the surface ∂B^3.
We compute the geodesic circle $S^1(q, R)$ explicitly. The conditions that the two points be on the same circle with center q and radius $R = \sqrt{(q,q)-1}$ in a plane containing $\xi = 0$ are

$$q = \mu_1 \xi + \mu_2 \eta,\ 2(q,\xi) = (\xi,\xi)+1,\ 2(q,\eta) = (\eta,\eta)+1, . \tag{25}$$

Solving these linear equations for μ_1, μ_2 yields

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \frac{1}{2((\xi,\xi)(\eta,\eta)-(\xi,\eta)^2)} \begin{bmatrix} (\eta,\eta) & -(\xi,\eta) \\ -(\xi,\eta) & (\xi,\xi) \end{bmatrix} \begin{bmatrix} (\xi,\xi)+1 \\ (\eta,\eta)+1 \end{bmatrix}. \tag{26}$$

4.3 Geodesics between preimage and image points on orbit surfaces.

First we compute the geodesic distance of the preimage and image points under class representatives eq. 1 from their scalar products.
7 Lemma: The length of all geodesic sections for fixed hyperbolic $g_0 = g_0(c, \gamma)$ and orbit parameter ρ is given by

$$\begin{aligned} &\langle L(g_0(-c/2, -\gamma/2)x(\rho,\phi), L(g_0(c/2,\gamma/2)x(\rho,\phi)\rangle \\ &= \cosh(2c) + (\cosh(2c) - \cos(2\gamma))(\sinh(2\rho))^2 \geq \cosh(2c). \end{aligned} \tag{27}$$

The minimal geodesic length under $L(g_0(c, \gamma))$ is reached for points on the straight geodesic orbit line with representative point $\rho = 0,\ x = (1, 0, 0, 0)$.
Proof: The scalar product between the points $\lambda = \pm 1$ on the orbit line by evaluation of eq. 21 yields eq. 27. This expression depends on the parameters c, γ of g_0 and on the parameter ρ which characterizes the orbit but is independent of the parameters λ, ϕ of the starting point on the orbit. The geodesic distance takes its minimum value $\cosh(2c)$ for $\rho = 0$. Its length is determined by the character $\chi(g_0)$ eq. 2.
Next we characterize the geodesics between the preimage and image points. We particularize the general construction of lemma 6 to the geodesic between $\xi(\lambda),\ \lambda = -1, +1$. Both vectors have the same length. From eqs. 23, 25 it can be shown that this geodesic is a section on the circle $S^1(q(-1, +1), R(-1, +1))$ with center and radius

$$\begin{aligned} q(-1,+1) &= \operatorname{cotanh}(2\rho)\frac{\cosh(c)}{\cos(\gamma)}(\cos(2\phi), \sin(2\phi), 0), \\ R(-1,+1) &= \sqrt{(q(-1,+1)), q(-1,+1)) - 1}. \end{aligned} \tag{28}$$

In Fig. 1 we show the orbit lines and geodesic sections for the Lorentz transformation $C_1 = L(g_0(c, \gamma))$ which is one generator of the uniform lattice $\Gamma(M)$ studied in section

6. For the orbit parameters we use six values $\mathrm{arccosh}(2\rho) = (0.0, 0.2, 0.4, 0.6, 0.8, 1.0)$, $\phi = 0$. In Fig. 2 we show the orbit and geodesic lines between two vertices of the Weber-Seifert dodecahedron analyzed in sections 5,6.

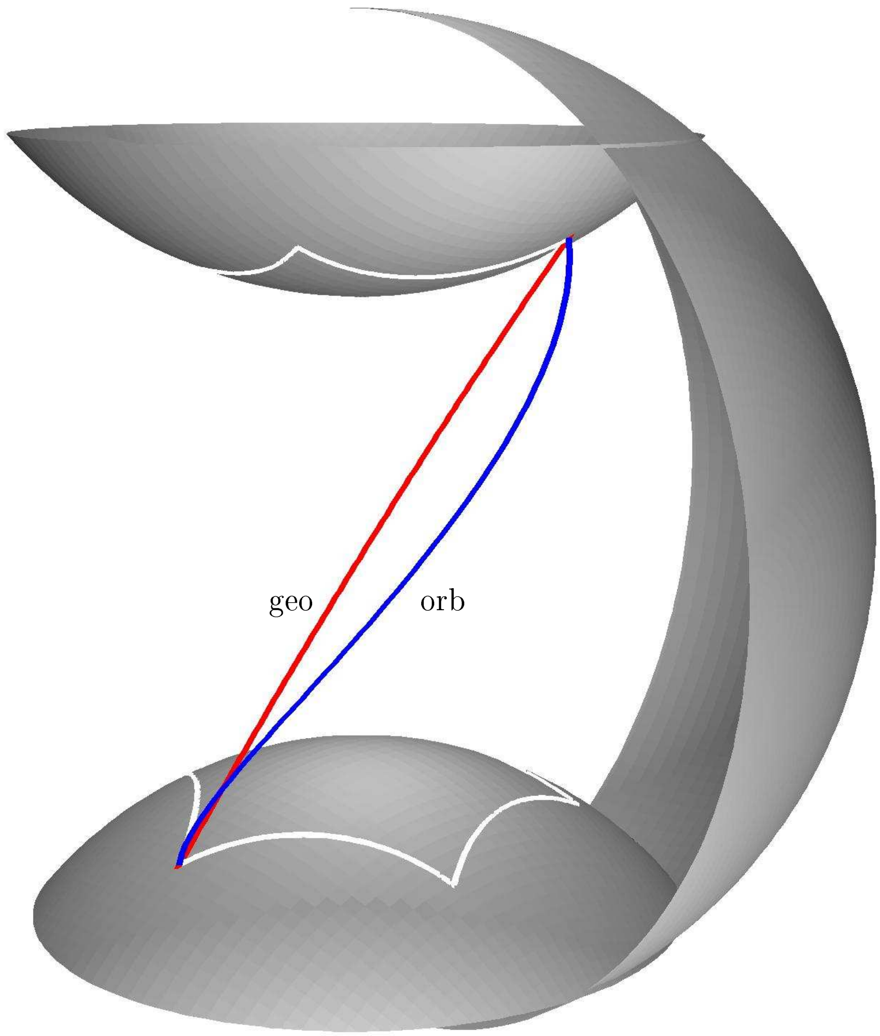

Fig. 2. View of part of B^3 and the same two Möbius spheres as in Fig. 1. An orbit line (orb) connects vertices of opposite pentagonal faces of the Weber-Seifert dodecahedron, the geodesic section between them is marked (geo).

4.4 Defect angle of geodesic loops.

Let a geodesic loop start at a point $P \in M$ and intersect itself at P. Denote by Δ the defect angle between the starting geodesic and its continuation after return

and intersection. For a smooth geodesic loop we must have $\Delta = 0$. We compute Δ on the universal covering B^3 by considering along with P the preimage and image $g_0^{-1}P, g_0P$ under a fixed diagonal element $g_0 \in \Gamma(M)$ eq. 1 with parameters c, γ. As orbit representative P we choose on the plane $\xi_3 = 0$ from eq. 23 the point $\xi(\lambda) \in B^3, \lambda = 0$ with $\phi = 0$. The images of this plane under g_0^{-1} and g_0 are two Möbius spheres similar to the ones used in eq. 24 but now for the parameter values $\lambda = \mp 2$. Consider the geodesic which runs on B^3 from $\xi(0)$ to $\xi(2) = g_0\xi(0)$. When mapped to the compact manifold M with $g_0 \in \Gamma(M)$, the geodesic intersects itself at $\xi(0)$. To compare the directions at start and after intersection, we apply g_0^{-1}. The image of the geodesic now runs from $\xi(-2)$ to the intersection at $\xi(0)$ and so corresponds to $\lambda = (-2, 0)$. The map g_0 is conformal, and so we can measure the defect angle on B^3 between the directions of the starting geodesic $\lambda = (0, 2)$ and the continuation of the geodesic $\lambda = (-2, 0)$ at $\xi(0)$.
With the abbreviations

$$
\begin{aligned}
&u := \cosh(2c),\ v := \sinh(2c),\ c := \cos(2\gamma),\ s := \sin(2\gamma), \\
&\tau := \cosh(2\rho),\ \sigma := \sinh(2\rho),
\end{aligned}
\tag{29}
$$

we compute from eq. 23 the three points $\xi(-2), \xi(0), \xi(2)$ in terms of the parameters of eq. 29,

$$
\xi(-2) = (1+\tau u)^{-1} \begin{bmatrix} \sigma c \\ \sigma s \\ -\tau v \end{bmatrix},\ \xi(0) = (1+\tau)^{-1} \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix},\ \xi(2) = (1+\tau u)^{-1} \begin{bmatrix} \sigma c \\ -\sigma s \\ \tau v \end{bmatrix}.
\tag{30}
$$

All three points are on a single orbit of the continuous normalizer. We observe from eq. 30 that, by the planar rotation

$$
\begin{aligned}
&R_{2,3} = \begin{bmatrix} c' & s' \\ -s' & c' \end{bmatrix}, \\
&c' := \tau v/\omega,\ s' := \sigma s/\omega,\ \omega := \sqrt{\sigma^2 s^2 + \tau^2 v^2},
\end{aligned}
\tag{31}
$$

applied to the components $2, 3$ of all three vectors, their new coordinate expressions become

$$
\xi(-2) = (1+\tau u)^{-1} \begin{bmatrix} \sigma c \\ 0 \\ -\omega \end{bmatrix},\ \xi(0) = (1+\tau)^{-1} \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix},\ \xi(2) = (1+\tau u)^{-1} \begin{bmatrix} \sigma c \\ 0 \\ \omega \end{bmatrix},
\tag{32}
$$

and so all three vectors are in the single new $1, 3$ plane of B^3.
We construct from eq. 25 the two geodesics which run between the pairs of points corresponding to $\lambda = (0, 2)$ and $\lambda = (-2, 0)$ respectively. Clearly both geodesic sections run on the new $1, 3$ plane and intersect at $\xi(0)$.

Evaluating eqs. 25, 26 for pairs of points we find for the center vectors $q(0,2),\ q(-2,0)$ of the two geodesic circles

$$\begin{aligned}
&q(0,2) = \mu_2\xi(0) + \mu_1\xi(2), \\
&q(-2,0) = \mu_1\xi(-2) + \mu_2\xi(0), \\
&\mu_1 = (1+\tau u)\frac{\tau(u-c)}{\omega^2},\ \mu_2 = (1+\tau)\frac{\tau[-\sigma^2(cu-1)+\tau^2 v^2]}{\sigma^2\omega^2}.
\end{aligned} \tag{33}$$

The radius vectors $\xi(0) - q(0,2), \xi(0) - q(-2,0)$ are perpendicular respectively to the starting and returning geodesics at $\xi(0)$. Therefore their angle of intersection determines the defect angle Δ. From eqs. 32, 33 we find in the new coordinates

$$\xi(0) - q(-2,0) = \begin{bmatrix} -\sigma^{-1} \\ 0 \\ \tau(u-c)/\omega \end{bmatrix},\ \xi(0) - q(0,2) = \begin{bmatrix} -\sigma^{-1} \\ 0 \\ -\tau(u-c)/\omega \end{bmatrix}. \tag{34}$$

The two vectors differ only by a reflection in the new 3-coordinate axis and so we get for half the defect angle the expression

$$\begin{aligned}
\tan(\frac{1}{2}\Delta) &= \frac{(\xi(0) - q(0,2))_3}{(\xi(0) - q(0,2))_1} \\
&= \sigma\frac{\tau(u-c)}{\omega} \\
&= \sinh(2\rho)\frac{\cosh(2\rho)(\cosh(2c) - \cos(2\gamma))}{\sqrt{\sinh(2\rho)^2\sin(2\gamma)^2 + \cosh(2\rho)^2\sinh(2c)^2}}.
\end{aligned} \tag{35}$$

In the final expression we replaced the abbreviations from eq. 29. We analyze the terms in the last expression. The exponential parameters for a fixed non-trivial hyperbolic g_0 must obey $(\cosh(2c) - \cos(2\gamma)) > 0,\ |\sinh(2c)| > 0$. Therefore the factor of $\sinh(2\rho)$ in eq. 35 is always different from zero. The value $\Delta = 0$ of the defect angle for a smooth geodesic loop enforces the unique orbit line with representative $\sinh(2\rho) = 0 \to \rho = 0$.

8 Lemma: The defect angle Δ for geodesic loops on M associated with g_0 is given as the function of group and orbit parameters by eq. 35. The only smooth geodesic loops associated with g_0 are sections of hyperbolic length $2c$ fixed by $\chi(g_0)$. They run on the infinite geodesic line $\xi = (0,0,\xi_3), -1 \leq \xi_3 \leq 1$, mapped from B^3 to M.

4.5 Orbit surfaces and geodesics for general Lorentz transformations.

So far we dealt only with the action of a class representative of a Lorentz transformation on general points of H^3. The general Lorentz transformation is given from

eq. 4 by conjugation with g_2g_1 as $L(g) = L((g_2g_1)g_0(g_2g_1)^{-1})$. Let c, γ again denote the exponential parameters of g_0.
By conjugation with g_2g_1 we introduce the two-parameter commutative general normalizer $N_g := (g_2g_1)N_{g_0}(g_2g_1)^{-1}$ which is conjugate to $SO(1,1,R) \times SO(2,R)$ and commutes with g. We let this normalizer act on H^3 or B^3 and obtain orbit surfaces. Lemma 4 is easily generalized to these orbit surfaces. The geometric results given in Lemma 5-8 for $L(g_0)$ can be transcribed to $L(g)$ if before we pass on H^3, B^3 from initial coordinates x to new ones defined as $y = L((g_2g_1)^{-1})x$. For general $L(g)$ it follows from lemma 7 that the shortest geodesic loop under $L(g)$ are sections of length fixed by the character $\chi(g)$ on the image of a straight infinite geodesic line under the inverse map $L(g_2g_1)$ acting on the coordinates y.
This action is isometric and conformal. Therefore the expressions eqs. 27 for the geodesic length and eq. 35 for the defect angle remain true in terms of the parameters for the diagonal form g_0 of g. For given g there is a unique set of shortest and smooth geodesics.

5 The hyperbolic Coxeter group.

The Weber-Seifert manifold is related to a hyperbolic Coxeter group. It generates the dodecahedral tesselation and has the uniform lattice $\gamma(M)$ as a subgroup. The hyperbolic Coxeter group produces discrete symmetries of the Weber-Seifert manifold M. Symmetries of compact manifolds play an important part in their classification, see [7] pp. 266-279.

5.1 The Coxeter group on $M(1,3)$ and B^3.

There is a hyperbolic Coxeter group which has the uniform lattice $\Gamma(M)$ of the Weber-Seifert space as a subgroup. This Coxeter group [10] p. 284 has the Coxeter diagram
$\circ \overset{5}{-} \circ \overset{3}{-} \circ \overset{5}{-} \circ$. Its four generators $R_1, \dots, R_4$ have the relations

$$\begin{aligned} & R_1^2 = R_2^2 = R_3^2 = R_4^2 = e, \\ & (R_1R_2)^5 = (R_2R_3)^3 = (R_3R_4)^5 = e, \\ & R_1R_3 = R_3R_1,\ R_1R_4 = R_4R_1,\ R_2R_4 = R_4R_2. \end{aligned} \tag{36}$$

The defining representation of this Coxeter group has a fundamental simplex in $M(1,3)$. Its boundaries are perpendicular to four Weyl unit vectors $a_1, \dots, a_4$ which generate the four reflections $R_1, \dots, R_4$. The dihedral angle between pairs of Weyl reflection hyperplanes is related to the exponents $m_{12} = 5, m_{23} = 3, m_{34} = 5$ in the second line of eq. 36 by

$$\langle a_i, a_{i+1} \rangle = \cos(\frac{\pi}{m_{i,i+1}}). \tag{37}$$

A set of space-like unit Weyl vectors in $M(1,3)$ for $\circ \overset{5}{—} \circ \overset{3}{—} \circ \overset{5}{—} \circ$ is found as:

$$\begin{aligned} a_1 &= (0,0,1,0), \\ a_2 &= (0, \frac{1}{2}\sqrt{-\tau+3}, -\frac{1}{2}\tau, 0), \\ a_3 &= (0, -\sqrt{\frac{\tau+2}{5}}, 0, -\sqrt{\frac{-\tau+3}{5}}), \\ a_4 &= (\frac{1}{2}\sqrt{4\tau-1}, 0, 0, \frac{1}{2}\sqrt{4\tau+3}). \end{aligned} \tag{38}$$

The Coxeter group acts on the hyperbolic space H^3 and on the conformal ball B^3. In the conformal ball model we place the intersection of the first three Weyl hyperplanes planes at $\xi = 0$. The first three Weyl reflection hyperplanes in B^3 become planes (spheres of infinite radius) perpendicular to the space part of the Weyl vectors in eq. 38. The fourth Weyl hyperplane becomes a Möbius sphere $S^2(q, R)$ which from eqs. 15, 38 has

$$q = (0, 0, \sqrt{\frac{4\tau+3}{4\tau-1}}),\ R = \frac{2}{\sqrt{4\tau-1}}. \tag{39}$$

The icosahedral Coxeter group $\circ \overset{\mathbf{5}}{—} \circ \overset{\mathbf{3}}{—} \circ$ is generated by R_1, R_2, R_3. In B^3 this subgroup generates from the fundamental simplex the dodecahedron of Weber and Seifert [13]. From the Coxeter group action it is formed by 120 copies of the fundamental Coxeter simplex. The unit vectors along the first three edge lines and axes of the fundamental simplex in B^3 are

$$\begin{aligned} e(\mathbf{5}) &= (0,0,1), \\ e(\mathbf{2}) &= (-\sqrt{\frac{-\tau+3}{5}}, 0, \sqrt{\frac{\tau+2}{5}}), \\ e(\mathbf{3}) &= (-\sqrt{\frac{\tau+2}{3\cdot 5}}, -\frac{-\tau+3}{\sqrt{3\cdot 5}}, \sqrt{\frac{4\tau+3}{3\cdot 5}}). \end{aligned} \tag{40}$$

The 12 outer faces of the dodecahedron are spherical pentagons on 12 spheres $S^2(q, R)$ whose center vectors q are directed along the 12 5fold axes of the dodecahedron, see Fig. 2. Inclusion of the generator R_4 generates a simplex tesselation of B^3. Sets of 120 simplices form the tiles of the dodecahedral tesselation of B^3.

5.2 Preimage of the Coxeter group associated with $Sl(2, C)$

In subsection 5.1 we gave on Minkowski space $M(1,3)$ the Weyl vectors which generate the hyperbolic Coxeter group and in subsection 6.2 the uniform lattice $\Gamma(M)$ of the Weber-Seifert hyperbolic dodecahedral manifold. For functions on $M(1,3)$ and on the coset space H^3 it seems natural to include two-component spinor states

χ_1, χ_2 whose components transform according to $g \in Sl(2,C)$ while the functional dependence follows $L(g)$,

$$T_g \begin{bmatrix} \chi_1(x) \\ \chi_2(x) \end{bmatrix} = g \begin{bmatrix} \chi_1(L(g^{-1}x)) \\ \chi_2(L(g^{-1}x)) \end{bmatrix}. \tag{41}$$

The operators eq. 41 form group homomorphisms,

$$T_{g_1} T_{g_2} = T_{g_1 g_2}. \tag{42}$$

To handle such spinor states we wish to extend $Sl(2,C)$ by preimages of Weyl reflections. The action of W_{e_2} on $M(1,3)$ can be expressed with respect to the matrix eq. 6 by $\tilde{x} \to \overline{\tilde{x}}$. Under $Sl(2,C)$ we find $g : \overline{\tilde{x}} \to \overline{g\ \tilde{x} g^\dagger}$. Therefore we associate to the Weyl reflection W_{e_2} as preimage the automorphism $C : g \to Cg = \overline{g}$. In line with eq. 10 we extend this operator by conjugation to $g \circ C \circ g^{-1}$. In operator products we use $C \circ g = \overline{g} \circ C$. First we find the preimage of the Weyl operator W_{e_3} as

$$\begin{aligned} g(e_3) \circ C \circ g(e_3)^{-1} &= (g(e_3)\overline{g(e_3)}^{-1}) \circ C \\ &= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \circ C, \\ g(e_3) &= \sqrt{\frac{1}{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}. \end{aligned} \tag{43}$$

Next for any Weyl vector a we can find a Lorentz transformation $L(l(a))$, $l(a) \in Sl(2,C) :\ a = L(l(a))e_3$. Then by extension of eq. 43 we get as preimage of the Weyl reflection W_a by use of eqs. 9, 10 the operator

$$\begin{aligned} s &= l(a)g(e_3) \circ C \circ (l(a)g(e_3))^{-1} \\ &= l(a) \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \overline{l(a)}^{-1} \circ C. \end{aligned} \tag{44}$$

We now apply these results and eq. 44 to find preimages s_1, s_2, s_3, s_4 associated with $Sl(2,C)$ for the generators of the Coxeter group. We find

$$\begin{aligned} s_1 &= -e \circ C, \\ s_2 &= \begin{bmatrix} \exp(-i\pi/5) & 0 \\ 0 & \exp(i\pi/5) \end{bmatrix} \circ C, \\ s_3 &= i \begin{bmatrix} \sqrt{\frac{\tau+2}{5}} & -\sqrt{\frac{-\tau+3}{5}} \\ -\sqrt{\frac{-\tau+3}{5}} & -\sqrt{\frac{\tau+2}{5}} \end{bmatrix} \circ C, \\ s_4 &= i \begin{bmatrix} 0 & \frac{1}{2}(\sqrt{4\tau+3} + \sqrt{4\tau-1}) \\ \frac{1}{2}(\sqrt{4\tau+3} - \sqrt{4\tau-1}) & 0 \end{bmatrix} \circ C. \end{aligned} \tag{45}$$

From these expressions we compute the products $s_i \circ s_j$ and find

$$\begin{aligned}
s_1 \circ s_1 &= s_2 \circ s_2 = s_3 \circ s_3 = s_4 \circ s_4 = e, \\
s_1 \circ s_2 &= \begin{bmatrix} -\exp(i\pi/5) & 0 \\ 0 & -\exp(-i\pi/5) \end{bmatrix}, \; (s_1 \circ s_2)^5 = e, \\
s_2 \circ s_3 &= i \begin{bmatrix} -\sqrt{\frac{\tau+2}{5}}\exp(-i\pi/5) & \sqrt{\frac{-\tau+3}{5}}\exp(-i\pi/5) \\ \sqrt{\frac{-\tau+3}{5}}\exp(i\pi/5) & \sqrt{\frac{\tau+2}{5}}\exp(i\pi/5) \end{bmatrix}, \\
\frac{1}{2}\mathrm{Tr}(s_2 \circ s_3) &= -\frac{1}{2} = \cos(2\pi/3), \; (s_2 \circ s_3)^3 = e, \\
s_3 \circ s_4 &= \begin{bmatrix} -\sqrt{\frac{-\tau+3}{5}}\frac{1}{2}(\sqrt{4\tau+3} - \sqrt{4\tau-1}) & \sqrt{\frac{\tau+2}{5}}\frac{1}{2}(\sqrt{4\tau+3} + \sqrt{4\tau-1}) \\ -\sqrt{\frac{\tau+2}{5}}\frac{1}{2}(\sqrt{4\tau+3} - \sqrt{4\tau-1}) & -\sqrt{\frac{-\tau+3}{5}}\frac{1}{2}(\sqrt{4\tau+3} + \sqrt{4\tau-1}) \end{bmatrix}, \\
\frac{1}{2}\mathrm{Tr}(s_3 \circ s_4) &= -\frac{1}{2}\tau = \cos(4\pi/5), \; (s_3 \circ s_4)^5 = e, \\
s_1 \circ s_3 &= - s_3 \circ s_1, \; s_1 \circ s_4 = -s_4 \circ s_1, \; s_2 \circ s_4 = -s_4 \circ s_2.
\end{aligned} \tag{46}$$

We use the trace relation eq. 2 to determine the class parameters of some products. All the products eq. 46 of two generators are elements of $Sl(2, C)$. By use of $L(g) = L(-g)$ one finds that all relations eq. 46 between the preimages $s_i, \; i = 1, \ldots, 4$ map correctly into the relations eq. 36 of the Coxeter group. We call the group generated by $\langle s_1, s_2, s_3, s_4 \rangle$ the preimage of the Coxeter group $\circ \overset{5}{-} \circ \overset{3}{-} \circ \overset{5}{-} \circ$.

6 Homotopy, uniform lattice $\Gamma(M)$, and homology groups of the hyperbolic dodecahedral space of Weber and Seifert.

6.1 The universal covering and its tesselation.

Weber and Seifert [13] describe their closed hyperbolic dodecahedral space M as follows: Any face of the dodecahedron is glued to its opposite face after a rotation by the angle $3\pi/5$. The universal covering of M must be a dodecahedral tesselation of H^3. The tesselation condition at any vertex of the fundamental dodecahedron enforces on H^3 the dodecahedral tesselation found from the Coxeter group. The uniform lattice $\Gamma(M)$ acts on the universal covering B^3 by isometries and generates the dodecahedral tesselation. $\Gamma(M)$ must act without fixpoints and therefore must be a proper subgroup of the Coxeter group $\circ \overset{5}{-} \circ \overset{3}{-} \circ \overset{5}{-} \circ$.

6.2 The uniform lattice $\Gamma(M)$.

Following the description of Weber and Seifert we construct a first generator C_1 of $\Gamma(M)$. We enumerate six dodecahedral faces by $i = 1, \ldots, 6$ and their opposites by $\bar{i}$. We choose face 1 perpendicular to e_3 in B^3. The neighbour faces of face 1 we enumerate counterclockwise as $2, 3, 4, 5, 6$. The relation of [13] Fig.1 to the present notation is

$$\begin{array}{lcccccc} [13]: & A & B & C & D & E & F \\ \text{present}: & 1 & 5 & 6 & 2 & 3 & 4 \end{array} \tag{47}$$

The action of the icosahedral Coxeter group from eqs. 36, 47 can now be given by signed permutations in cycle form. The operation that moves face $\bar{1}$ to face 1 is a Lorentz transformation in the $(0, 3)$ plane of $M(1, 3)$ which commutes with a rotation by $2\tilde{\gamma} = 3\pi/5$ in the $(1, 2)$ plane, see eq. 8. The parameter $2\tilde{c}$ of the Lorentz transformation is found from the position of the Weyl hyperplane perpendicular to a_4 eq. 38. With this input and with eq. 8 the generator C_1 is given as

$$C_1 = L(g_0(\tilde{c}, \tilde{\gamma})),\ \tilde{c} = \operatorname{arccosh}(\frac{1}{2}\sqrt{4\tau + 3}),\ \tilde{\gamma} = 3\pi/10. \tag{48}$$

All other generators are conjugates of C_1 under the icosahedral Coxeter group and may be denoted as $C_i,\ i = 1, \ldots, 6$. Generators for opposite faces are inverses and so

$$(C_i)^{-1} = C_{\bar{i}}. \tag{49}$$

We are left with 6 generators C_i of $\Gamma(M)$. We shall employ the Coxeter group to find the relations between these generators. The first three Coxeter generators in the signed cycle notation read

$$R_1 = (23)(46),\ R_2 = (24)(56),\ R_3 = (15)(2\bar{3}). \tag{50}$$

Moreover we introduce for the 5fold rotation in the direction i the symbol $5_i,\ (5_i)^5 = e$. In the cycle notation given above we have $5_1 = (23456)$. We also introduce the parity $P := (1\bar{1})(2\bar{2})(3\bar{3})(4\bar{4})(5\bar{5})(6\bar{6})$ which commutes with all elements of the icosahedral Coxeter group. Then we find the following relation between the Coxeter generator R_4 and the generator C_1 of $\Gamma(M)$:

$$R_4 = C_1\ P\ 5_1. \tag{51}$$

As we have expressed generators of the Coxeter group by generators of $\Gamma(M)$, we may rewrite relations of the Coxeter group in terms of generators of $\Gamma(M)$. We find

$$\begin{aligned} (R_3R_4) &= C_5\ 5_2^{-2},\ (R_3R_4)^2 = C_5\ C_{\bar{6}}\ 5_2^{-4}, \\ (R_3R_4)^3 &= C_5\ C_{\bar{6}}C_{\bar{3}}\ 5_2^{-6},\ (R_3R_4)^4 = C_5\ C_{\bar{6}}C_{\bar{3}}C_4\ 5_2^{-8}, \\ (R_3R_4)^5 &= C_5\ C_{\bar{6}}C_{\bar{3}}C_4C_{\bar{1}} = e. \end{aligned} \tag{52}$$

In the last step we applied $(5_2)^{10} = e$ and a relation of the Coxeter group from eq. 36. The geometric origin of the relation between generators of $\Gamma(M)$ can now be seen: The powers of (R_3R_4) generate in B^3 5fold rotations around a fixed edge of the fundamental simplex and of the fundamental dodecahedron. The relations eq. 52 transcribe these reflection-generated rotations into relations between the generators of the uniform lattice $\Gamma(M)$. It is now easy to find the other relations. According to [13] there are six sets of equivalent edge lines in the fundamental dodecahedron. Each of these sets gives a relation between the generators. The six relations become

$$\begin{aligned} &C_5\, C_{\overline{6}}C_{\overline{3}}C_4C_{\overline{1}} = e,\ C_6C_{\overline{2}}C_{\overline{4}}C_5C_{\overline{1}} = e,\ C_2C_{\overline{3}}C_{\overline{5}}C_6C_{\overline{1}} = e, \\ &C_3C_{\overline{4}}C_{\overline{6}}C_2C_{\overline{1}} = e,\ C_4C_{\overline{5}}C_{\overline{2}}C_3C_{\overline{1}} = e,\ C_4C_6C_3C_5C_2 = e. \end{aligned} \tag{53}$$

9 Lemma: The uniform lattice $\Gamma(M)$ of the Weber Seifert hyperbolic dodecahedral manifold M is generated by $\langle C_1, C_2, C_3, C_4, C_5, C_6\rangle$ with the relations eq. 53. These relations are associated with 6 edges a, b, c, d, e, f of the dodecahedron in [13] Fig.1.

6.3 $\Gamma(M)$ and the icosahedral Coxeter group.

10 Lemma: The uniform lattice $\Gamma(M)$ of the Seifert-Weber dodecahedral hyperbolic space forms a semidirect product with the icosahedral Coxeter group $\circ\overset{5}{-}\circ\overset{3}{-}\circ$,

$$(\Gamma(M)) \times_s (\circ\overset{5}{-}\circ\overset{3}{-}\circ). \tag{54}$$

This semidirect product is isomorphic to the full hyperbolic Coxeter group.
Proof: (i): Elements of the icosahedral Coxeter subgroup generated by R_1, R_2, R_3 clearly by conjugation map generators of $\Gamma(M)$ into generators. The elements of $\Gamma(M)$ have no fixpoint whereas any element of $\circ\overset{5}{-}\circ\overset{3}{-}\circ$ has the point $\xi = 0$ as fixpoint. Therefore the intersection obeys $(\Gamma(M)) \cap (\circ\overset{5}{-}\circ\overset{3}{-}\circ) = e$. These two properties suffice to show that the two groups form a semidirect product group eq. 54 with $\Gamma(M)$ the normal subgroup. (ii): The remaining generator R_4 of the full hyperbolic Coxeter group may be expressed from eq. 51 as an element of the semidirect product group. Since then all four generators eq. 36 of the full Coxeter group are in the semidirect product, it must be isomorphic to the full Coxeter group,

$$(\Gamma(M)) \times_s (\circ\overset{5}{-}\circ\overset{3}{-}\circ) \sim \circ\overset{5}{-}\circ\overset{3}{-}\circ\overset{5}{-}\circ\,. \tag{55}$$

The isomorphism eq. 54 yields a new interpretation of the group relations eq. 53 for $\Gamma(M)$. If the Coxeter group is rewritten as the semidirect product with the normal subgroup $\Gamma(M)$, the Coxeter relations enforce on $\Gamma(M)$ these group relations.
The semidirect product group eq. 54 is a natural hyperbolic counterpart of a Euclidean symmorphic space group, with $\Gamma(M)$ being a non-commutative version of

the translation group and $\circ \overset{5}{—} \circ \overset{3}{—} \circ$ of the point group. It follows that a preimage of $\Gamma(M)$ in $Sl(2,C)$ can be constructed from the generators given in section 5.2. The generator C_1 was expressed already in eq. 51 by an element $g_0(c,\gamma) \in Sl(2,C)$. All elements of $\Gamma(M)$ are even words in the generators of the Coxeter group with preimages in $Sl(2,C)$.
From eq. 54, the uniform lattice $\Gamma(M)$ of the Weber-Seifert manifold is a normal subgroup of the hyperbolic Coxeter group. A conjugation with an element of the Coxeter group maps elements of $\Gamma(M)$ into one another and therefore relates the corresponding sets of geodesic loops beyond $\Gamma(M)$.
The elements of $\Gamma(M)$ are words in the 6 generators C_i. These words must be analyzed in view of the 6 relations eq. 53. An alternative is to rewrite the words of $\Gamma(M)$ as new words in the 4 generators R_j. The relations eq. 36 between these generators are much simpler to control, see [5] p. 171, and so yield an efficient approach to the word problem. By the methods of section 5.2, this analysis can be carried out on the level of $Sl(2,C)$ without use of Lorentz transformations.

6.4 The homology group $H_1(M)$.

The homology group of the Weber-Seifert dodecahedral manifold M can be derived by abelianization of $\pi_1(M) \sim \Gamma(M)$. Applying abelianization to the generators and to their relations eq. 44 one obtains:
11 Lemma: The homology group $H_1(M)$ as abelianization of the homotopy group $\Gamma(M)$ is the direct product

$$H_1(M) = (Z/5Z) \times (Z/5Z) \times (Z/5Z). \tag{56}$$

of three cyclic groups of order 5. The icosahedral Coxeter group acts on these cyclic group with a modular representation by 3×3 matrices of integer entries modulo 5.
Proof: We apply abelianization to the defining relations eq. 53. The abelian image of a generator we denote by a label a. If we multiply the first five relations with one another and abelianize we find

$$(C_1^a)^5 = e. \tag{57}$$

For each generator C_i^a we can find 5 relations (or their inverses) proportional to C_i^a. Multiplying them yields the analog of equation eq. 57 for $i = 2, \ldots, 6$. It is possible to select 3 independent generators say C_1^a, C_4^a, C_5^a of order 5 and to express the other ones by them in the form of monomials,

$$\begin{aligned} C_j^a &= (C_1^a)^{m_{1j}} (C_4^a)^{m_{4j}} (C_5^a)^{m_{5j}},\ j = 2,3,6, \\ [m_1, m_4, m_5] &= \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 4 \\ 3 & 4 & 2 \end{bmatrix}. \end{aligned} \tag{58}$$

where all integers are taken modulo 5. We can now implement a conjugation action of the icosahedral Coxeter group. We write the conjugation of the three generators of $H_1(M)$ with the help of eq. 58 as

$$
\begin{aligned}
R_i(C_l^a)R_i^{-1} &= (C_1^a)^{m^i_{1l}}(C_4^a)^{m^i_{4l}}(C_5^a)^{m^i_{5l}}, \qquad (59)\\
m^1 &= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 2 & 1 \end{bmatrix},\ m^2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 3 & 2 \end{bmatrix},\ m^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ l = 1, 4, 5.
\end{aligned}
$$

The entries of all matrices in eqs. 58, 59 are taken modulo 5. The three modular matrices m^1, m^2, m^3 obey $(m^l)^2 = 1_3,\ l = 1, 2, 3$. They generate a modular representation of the icosahedral Coxeter group. Modular representations of the symmetric group are discussed in [11].

6.5 Conclusion: geodesic loops on M.

Our main results on geodesic loops are:
(1) Any $g \in \Gamma(M) \subset Sl(2, C)$ produces a variety of geodesic loops. The character $\chi(g)$ determines the class parameters, the matrix (g_2g_1) of eq. 4 that transforms g to diagonal form g_0 yields the in-class parameters of g.
(2) The continuous commutative normalizer $N_g \subset Sl(2, C)$ of the discrete subgroup generated by g determines orbits on B^3. Geodesic loops start and end on the same orbit. Their length is given by eq. 27, the defect angle at their self-intersection by eq. 35 in terms of the class and orbit parameters.
(3) For any g there is a unique set of shortest and smooth geodesic loops. All of them are sections on a single infinite geodesic line on B^3.
(4) For elements in the same class of $\Gamma(M)$, the in-class parameters of the diagonalizing matrix (g_2g_1) determine the normalizer N_g as a subgroup conjugate to N_{g_0}, the conjugate orbits, and the single infinite geodesic line for the shortest and smooth geodesic loops.
(5) To compare geodesics arising from different classes of $\Gamma(M)$ one must study the words in this group. Symmetries, exemplified by the Weber-Seifert dodecahedral manifold, yield additional relations between geodesic loops. For the Weber-Seifert manifold, the words can be rewritten in terms of generators for the hyperbolic Coxeter group. Since the latter group has the simpler relations eq. 36, the word problem is simplified by this rewriting.

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